

Explicit laws of large numbers for random nearest-neighbour type graphs

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Abstract

Under the unifying umbrella of a general result of Penrose & Yukich [*Ann. Appl. Probab.*, (2003) **13**, 277–303] we give laws of large numbers (in the L^p sense) for the total power-weighted length of several nearest-neighbour type graphs on random point sets in \mathbf{R}^d , $d \in \mathbf{N}$. Some of these results are known; some are new. We give limiting constants explicitly, where previously they have been evaluated in less generality or not at all. The graphs we consider include the k -nearest neighbours graph, the Gabriel graph, the minimal directed spanning forest, and the on-line nearest-neighbour graph.

Key words and phrases: Nearest-neighbour type graphs; laws of large numbers; spanning forest; spatial network evolution.

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1 Introduction

Graphs constructed on random point sets in \mathbf{R}^d ($d \in \mathbf{N}$), formed by joining nearby points according to some deterministic rule, have recently received considerable interest [20, 29, 31]. Such graphs include the geometric graph, the minimal spanning tree, and (as studied in this paper) the nearest-neighbour graph and its relatives. Applications include the modelling of spatial networks, as well as statistical procedures.

The graphs in this paper are based on edges between nearest neighbours, sometimes in some restricted sense. A unifying characteristic of these graphs is *stabilization*: roughly speaking, the configuration of edges around any particular vertex is not affected by changes to the vertex set outside of some sufficiently large (but finite) ball. Thus these graphs are locally determined in some sense.

A functional of particular interest is the total edge length of the graph, or, more generally, the total power-weighted edge length (i.e. the sum of the edge lengths each

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raised to a given power $\alpha \geq 0$). The large-sample asymptotic theory for power-weighted length of stabilizing graphs is now well understood; see e.g. [15, 20, 21, 25, 26, 29, 31].

In the present paper we collect several laws of large numbers (LLNs) for total power-weighted length from the family of nearest-neighbour type graphs, defined on independent random points on \mathbf{R}^d . We present these results as corollaries to a general umbrella theorem of Penrose & Yukich [26]. Some of the results (for the most common graphs) are known to various extents in the literature; others are new. We take a unified approach which highlights the connections between these results.

In particular, all our results are explicit: we give explicit expressions for limiting constants. In some cases these constants have been seen previously in the literature.

Nearest-neighbour graphs and nearest-neighbour distances in \mathbf{R}^d are of interest in several areas of applied science, including the social sciences, geography and ecology, where proximity data are often important (see e.g. [16, 27]). Ad-hoc networks, in which nodes scattered in space are connected according to some geometric rule, are of interest with respect to various types of communication networks. Quantities of interest such as overall network throughput may be related to power-weighted length.

In the analysis of multivariate data, in particular via non-parametric statistics, nearest-neighbour graphs and near-neighbour distances have found many applications, including goodness of fit tests, classification, regression, noise estimation, density estimation, dimension identification, cluster analysis, and the two-sample and multi-sample problems; see for example [6, 7, 8, 10, 12, 13, 30] and references therein.

In this paper we give a new LLN for the total power-weighted length of the *on-line nearest-neighbour graph* (ONG), which is one of the simplest models of network evolution. We give a detailed description later. In the ONG on a sequence of points arriving in \mathbf{R}^d , each point after the first is joined by an edge to its nearest predecessor. The ONG appeared in [4] as a simple model for the evolution of the Internet graph. Figure 1 shows a sample realization of an ONG.

Recently, graphs with an ‘on-line’ structure, in which vertices are added one by one and connected to existing vertices via some rule, have been the subject of considerable study in relation to the modelling of real-world networks. The ONG is one of the simplest network evolution models that captures some of the observed characteristics of real-world networks, such as spatial structure and sequential growth.

We also consider the *minimal directed spanning forest* (MDSF). The MDSF is constructed on a partially ordered point set in \mathbf{R}^d by connecting each point to its nearest neighbour amongst those points (if any) that precede it in the partial order. If an MDSF is a tree, it is called a *minimal directed spanning tree* (MDST).

The MDST was introduced by Bhatt & Roy in [5] as a model for drainage or communications networks, in $d = 2$, with the ‘coordinatewise’ partial order \preceq^* , such that $(x_1, y_1) \preceq^* (x_2, y_2)$ iff $x_1 \leq x_2$ and $y_1 \leq y_2$. In this version of the MDSF, each point is joined by an edge to its nearest neighbour in its ‘south-westerly’ quadrant. In the present paper we give new LLNs for the total power-weighted length for a family of MDSFs indexed by partial orderings on \mathbf{R}^2 , which include \preceq^* as a special case. Figure 1 shows an example of a MDSF under \preceq^* .

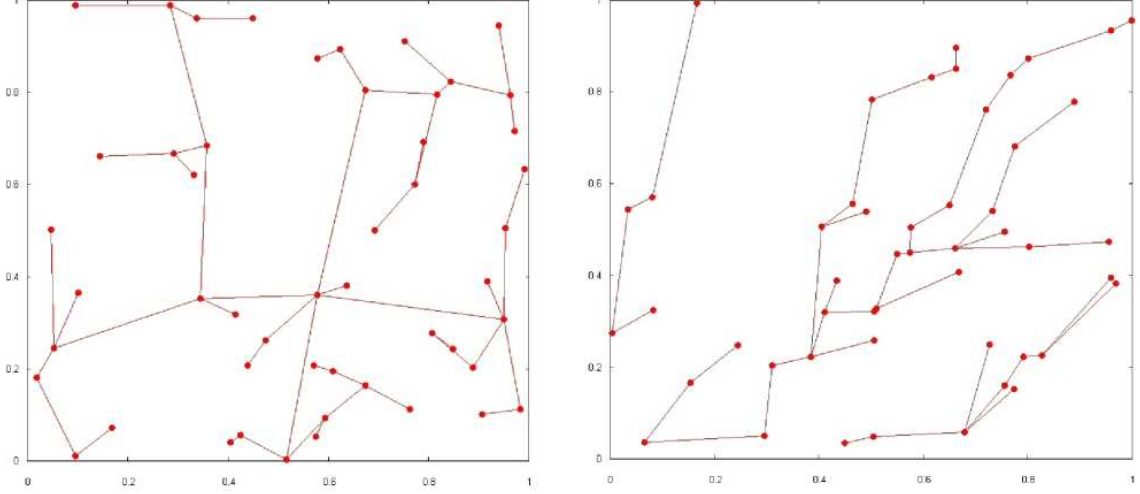


Figure 1: Realizations of the ONG (left) and MDSF under \preceq^* (right), each on 50 simulated uniform random points in the unit square.

2 Notation and results

Notions of *stabilizing* functionals of point sets have recently proved to be a useful basis for establishing limit theorems for functionals of random point sets in \mathbf{R}^d . In particular, Penrose & Yukich [25, 26] prove general central limit theorems and laws of large numbers for stabilizing functionals.

The LLNs we give in the present paper are all derived ultimately from Theorem 2.1 of [26], which we restate as Theorem 1 below before we present our results.

In order to describe the result of [26], we need to introduce some notation. Let $d \in \mathbf{N}$. Let $\|\cdot\|$ be the Euclidean norm on \mathbf{R}^d . Write $\text{card}(\mathcal{X})$ for the cardinality of a finite set $\mathcal{X} \subset \mathbf{R}^d$. For a locally finite point set $\mathcal{X} \subset \mathbf{R}^d$, $a > 0$, and $\mathbf{y} \in \mathbf{R}^d$, let $\mathbf{y} + a\mathcal{X}$ denote the set $\{\mathbf{y} + a\mathbf{x} : \mathbf{x} \in \mathcal{X}\}$. Let $B(\mathbf{x}; r)$ denote the closed Euclidean ball with centre $\mathbf{x} \in \mathbf{R}^d$ and radius $r > 0$. Let $\mathbf{0}$ denote the origin in \mathbf{R}^d .

Let $\xi(\mathbf{x}; \mathcal{X})$ be a measurable $[0, \infty)$ -valued function defined for all pairs $(\mathbf{x}, \mathcal{X})$, where $\mathcal{X} \subset \mathbf{R}^d$ is finite and $\mathbf{x} \in \mathcal{X}$. Assume ξ is translation invariant, that is, for all $\mathbf{y} \in \mathbf{R}^d$, $\xi(\mathbf{y} + \mathbf{x}; \mathbf{y} + \mathcal{X}) = \xi(\mathbf{x}; \mathcal{X})$. When $\mathbf{x} \notin \mathcal{X}$, we abbreviate the notation $\xi(\mathbf{x}; \mathcal{X} \cup \{\mathbf{x}\})$ to $\xi(\mathbf{x}; \mathcal{X})$. For our applications, ξ will be *homogeneous of order* $\alpha \geq 0$, that is $\xi(r\mathbf{x}; r\mathcal{X}) = r^\alpha \xi(\mathbf{x}; \mathcal{X})$ for all $r > 0$, all finite point sets \mathcal{X} , and all $\mathbf{x} \in \mathcal{X}$.

For any locally finite point set $\mathcal{X} \subset \mathbf{R}^d$ and any $\ell \in \mathbf{N}$ define

$$\begin{aligned} \xi^+(\mathcal{X}; \ell) &:= \sup_{k \in \mathbf{N}} \left(\text{ess sup} \left\{ \xi(\mathbf{0}; (\mathcal{X} \cap B(\mathbf{0}; \ell)) \cup \mathcal{A}^*) : \mathcal{A} \in (\mathbf{R}^d \setminus B(\mathbf{0}; \ell))^k \right\} \right), \text{ and} \\ \xi^-(\mathcal{X}; \ell) &:= \inf_{k \in \mathbf{N}} \left(\text{ess inf} \left\{ \xi(\mathbf{0}; (\mathcal{X} \cap B(\mathbf{0}; \ell)) \cup \mathcal{A}^*) : \mathcal{A} \in (\mathbf{R}^d \setminus B(\mathbf{0}; \ell))^k \right\} \right), \end{aligned}$$

where for $\mathcal{A} = (\mathbf{x}_1, \dots, \mathbf{x}_k) \in (\mathbf{R}^d)^k$ we put $\mathcal{A}^* = \{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ (provided all k vectors are distinct). Define the *limit* of ξ on \mathcal{X} by

$$\xi_\infty(\mathcal{X}) := \limsup_{\ell \rightarrow \infty} \xi^+(\mathcal{X}; \ell).$$

We say the functional ξ *stabilizes* on \mathcal{X} if

$$\lim_{\ell \rightarrow \infty} \xi^+(\mathcal{X}; \ell) = \lim_{\ell \rightarrow \infty} \xi^-(\mathcal{X}; \ell) = \xi_\infty(\mathcal{X}).$$

Stabilization can be interpreted loosely as the property that the value of the functional at a point is unaffected by changes in the configuration of points at a sufficiently large distance from that point.

Let f be a probability density function on \mathbf{R}^d . For $n \in \mathbf{N}$ let $\mathcal{X}_n := (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n)$ be the point process consisting of n independent random d -vectors with common density f . With probability one, \mathcal{X}_n has distinct inter-point distances; hence all the nearest-neighbour type graphs on \mathcal{X}_n that we consider are almost surely unique.

Let \mathcal{H}_1 be a homogeneous Poisson point process of unit intensity on \mathbf{R}^d . The following general LLN is due to Penrose & Yukich, and is obtained from Theorem 2.1 of [26] together with equation (2.9) there (the homogeneous case).

Theorem 1 *Let $q \in \{1, 2\}$. Suppose that ξ is homogeneous of order α and almost surely stabilizes on \mathcal{H}_1 , with limit $\xi_\infty(\mathcal{H}_1)$. If ξ satisfies the moments condition*

$$\sup_{n \in \mathbf{N}} E[\xi(n^{1/d} \mathbf{X}_1; n^{1/d} \mathcal{X}_n)^p] < \infty, \quad (1)$$

for some $p > q$, then as $n \rightarrow \infty$,

$$n^{-1} \sum_{\mathbf{x} \in \mathcal{X}_n} \xi(n^{1/d} \mathbf{x}; n^{1/d} \mathcal{X}_n) \xrightarrow{L^q} E[\xi_\infty(\mathcal{H}_1)] \int_{\text{supp}(f)} f(\mathbf{x})^{(d-\alpha)/d} d\mathbf{x},$$

and the limit is finite.

From this result we will derive LLNs for the total power-weighted length for a collection of nearest-neighbour type graphs. Let $j \in \mathbf{N}$. A point $\mathbf{x} \in \mathcal{X}$ has a j -th *nearest neighbour* $\mathbf{y} \in \mathcal{X} \setminus \{\mathbf{x}\}$ if $\text{card}(\{\mathbf{z} : \mathbf{z} \in \mathcal{X} \setminus \{\mathbf{x}\}, \|\mathbf{z} - \mathbf{x}\| < \|\mathbf{y} - \mathbf{x}\|\}) = j - 1$.

For all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^d$ we define the weight function

$$w_\alpha(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|^\alpha,$$

for some fixed parameter $\alpha \geq 0$. By the total power-weighted edge length of a graph with edge set E (where edges may be directed or undirected), we mean the functional

$$\sum_{(\mathbf{u}, \mathbf{v}) \in E} w_\alpha(\mathbf{u}, \mathbf{v}) = \sum_{(\mathbf{u}, \mathbf{v}) \in E} \|\mathbf{u} - \mathbf{v}\|^\alpha.$$

We will often assume one of the following conditions on the function f — either

(C1) f is supported by a convex polyhedron in \mathbf{R}^d and is bounded away from 0 and infinity on its support; or

(C2) for weight exponent $\alpha \in [0, d)$, we require that $\int_{\mathbf{R}^d} f(\mathbf{x})^{(d-\alpha)/d} d\mathbf{x} < \infty$ and $\int_{\mathbf{R}^d} \|\mathbf{x}\|^r f(\mathbf{x}) d\mathbf{x} < \infty$ for some $r > d/(d - \alpha)$.

In some cases, we take $f(\mathbf{x}) = 1$ for $\mathbf{x} \in (0, 1)^d$ and $f(\mathbf{x}) = 0$ otherwise, in which case we denote $\mathcal{X}_n = \mathcal{U}_n = (\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n)$, the binomial point process consisting of n independent uniform random vectors on $(0, 1)^d$.

In the remainder of this section we present our LLNs derived from Theorem 1. Theorems 2, 3, and 6 follow directly from Theorem 1 and results in [26], up to evaluation of constants, while Theorems 4 and 5 need some more work. These results are natural companions, as are their proofs, which we present in Section 3 below; in particular the proof of Theorem 2 is useful for the other proofs.

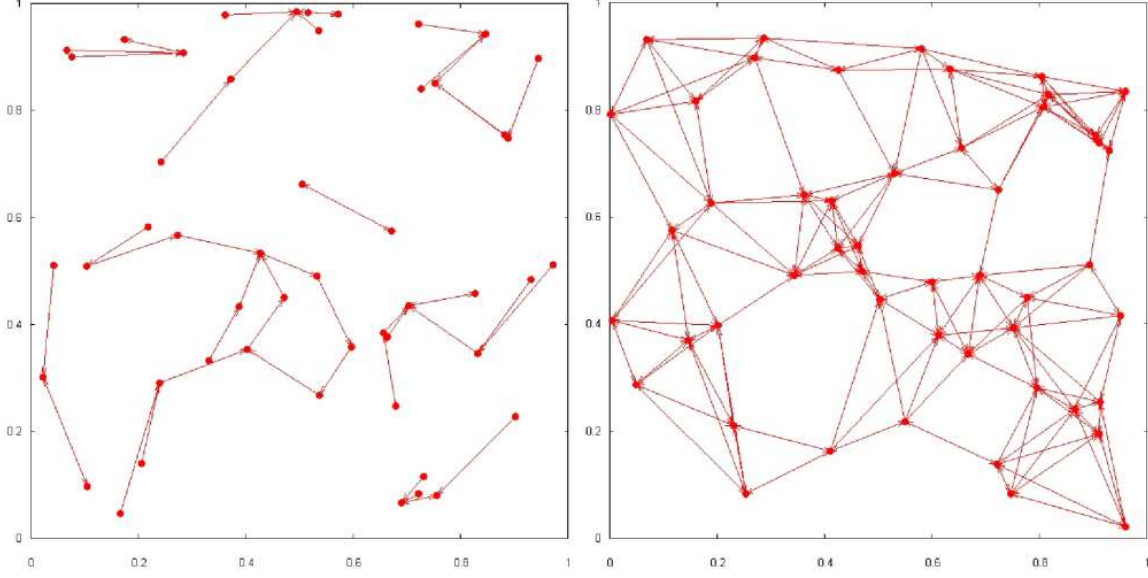


Figure 2: Realizations of the 3-rd NNG' (left) and 5-NNG' (right), each on 50 simulated uniform random points in the unit square.

2.1 The k -nearest neighbours and j -th nearest neighbour graphs

Let $j \in \mathbf{N}$. In the j -th nearest-neighbour (directed) graph on \mathcal{X} , denoted by j -th NNG'(\mathcal{X}), a directed edge joins each point of \mathcal{X} to its j -th nearest-neighbour.

Let $k \in \mathbf{N}$. In the k -nearest neighbours (directed) graph on \mathcal{X} , denoted k -NNG'(\mathcal{X}), a directed edge joins each point of \mathcal{X} to each of its first k nearest neighbours in \mathcal{X} (i.e. each of its j -th nearest neighbours for $j = 1, 2, \dots, k$). Clearly the 1-th NNG' and 1-NNG' coincide, giving the standard nearest-neighbour (directed) graph. See Figure 2 for realizations of particular j -th NNG', k -NNG'.

We also consider the k -nearest neighbours (undirected) graph on \mathcal{X} , denoted by k -NNG(\mathcal{X}), in which an undirected edge joins $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ if \mathbf{x} is one of the first k nearest neighbours of \mathbf{y} , or \mathbf{y} is one of the first k nearest neighbours of \mathbf{x} (or both).

From now on we take the point set \mathcal{X} to be *random*, in particular, for $n \in \mathbf{N}$, we take $\mathcal{X} = \mathcal{X}_n$. For $d \in \mathbf{N}$ and $\alpha \geq 0$, let $\mathcal{L}_j^{d,\alpha}(\mathcal{X}_n)$, $\mathcal{L}_{\leq k}^{d,\alpha}(\mathcal{X}_n)$ denote respectively the total power-weighted edge length of the j -th nearest-neighbour (directed) graph, k -nearest neighbours (directed) graph on $\mathcal{X}_n \subset \mathbf{R}^d$. Note that

$$\mathcal{L}_{\leq k}^{d,\alpha}(\mathcal{X}_n) = \sum_{j=1}^k \mathcal{L}_j^{d,\alpha}(\mathcal{X}_n). \quad (2)$$

For $d \in \mathbf{N}$, we denote the volume of the unit d -ball (see e.g. (6.50) in [14]) by

$$v_d := \pi^{d/2} [\Gamma(1 + (d/2))]^{-1}. \quad (3)$$

Theorems 2 and 4 below feature constants $C(d, \alpha, k)$ defined for $d, k \in \mathbf{N}$, $\alpha \geq 0$ by

$$C(d, \alpha, k) := v_d^{-\alpha/d} \frac{d}{d + \alpha} \frac{\Gamma(k + 1 + (\alpha/d))}{\Gamma(k)}. \quad (4)$$

Our first result is Theorem 2 below, which gives LLNs for $\mathcal{L}_j^{d,\alpha}(\mathcal{X}_n)$ and $\mathcal{L}_{\leq k}^{d,\alpha}(\mathcal{X}_n)$, with explicit expressions for the limiting constants; it is the natural starting point for our LLNs for nearest-neighbour type graphs. Let $\text{supp}(f)$ denote the support of f ; under (C1), $\text{supp}(f)$ is a convex polyhedron, under (C2) $\text{supp}(f)$ is \mathbf{R}^d .

Theorem 2 *Let $d \in \mathbf{N}$. The following results hold, with $p = 2$, for $\alpha \geq 0$ if f satisfies condition (C1), and, with $p = 1$, for $\alpha \in [0, d)$ if f satisfies condition (C2).*

(a) *For j -th NNG' on \mathbf{R}^d we have, as $n \rightarrow \infty$,*

$$n^{(\alpha-d)/d} \mathcal{L}_j^{d,\alpha}(\mathcal{X}_n) \xrightarrow{L^p} v_d^{-\alpha/d} \frac{\Gamma(j + (\alpha/d))}{\Gamma(j)} \int_{\text{supp}(f)} f(\mathbf{x})^{(d-\alpha)/d} d\mathbf{x}. \quad (5)$$

(b) *For k -NNG' on \mathbf{R}^d we have, as $n \rightarrow \infty$,*

$$n^{(\alpha-d)/d} \mathcal{L}_{\leq k}^{d,\alpha}(\mathcal{X}_n) \xrightarrow{L^p} C(d, \alpha, k) \int_{\text{supp}(f)} f(\mathbf{x})^{(d-\alpha)/d} d\mathbf{x}. \quad (6)$$

In particular, as $n \rightarrow \infty$,

$$n^{(\alpha-d)/d} \mathcal{L}_{\leq k}^{d,\alpha}(\mathcal{U}_n) \xrightarrow{L^p} C(d, \alpha, k). \quad (7)$$

Remarks. (a) If we use a different norm on \mathbf{R}^d from the Euclidean, Theorem 2 remains valid with v_d redefined as the volume of the unit d -ball in the chosen norm.

(b) Theorem 2 is essentially contained in Theorem 2.4 of [26], with the constants evaluated explicitly. There are several related LLN results in the literature. Theorem 8.3 of [31] gives LLNs (with complete convergence) for $\mathcal{L}_{\leq k}^{d,1}(\mathcal{X}_n)$ (see also [17]); the limiting constants are not given. Avram & Bertsimas (Theorem 7 of [2]) state a result on the limiting expectation (and hence the constant in the LLN) for $\mathcal{L}_j^{2,1}(\mathcal{U}_n)$, which they attribute to Miles [18] (see also p. 101 of [31]). The constant in [2] is given as

$$\frac{1}{2} \pi^{-1/2} \sum_{i=1}^j \frac{\Gamma(i - (1/2))}{\Gamma(i)},$$

which simplifies (by induction on j) to $\pi^{-1/2} \Gamma(j + (1/2)) / \Gamma(j)$, the $d = 2$, $\alpha = 1$ case of (5) in the case $\mathcal{X}_n = \mathcal{U}_n$.

(c) Related results are the asymptotic expectations of j -th nearest neighbour distances in finite point sets given in [9] and [19]. The results in [19] are consistent with the $\alpha = 1$ case of our (7). The result in [9] includes general α and certain non-uniform densities, although their conditions on f are more restrictive than our (C1); the result is consistent with (6). Also, [9] gives (equation (6.4)) a weak LLN for the empirical mean k -nearest neighbour *distance*. With Theorem 2.4 of [26], the results in [9] yield LLNs for the total weight of the j -th NNG' and k -NNG' only when $d - 1 < \alpha < d$ (due to the rates of convergence given in [9]).

(d) Smith [28] gives, in some sense, expectations of randomly selected edge lengths for nearest-neighbour type graphs on the homogeneous Poisson point process of unit intensity in \mathbf{R}^d , including the j -th NNG', nearest-neighbour (undirected) graph, and Gabriel graph. His results coincide with ours only for the j -th NNG', since here each vertex contributes

a fixed number (j) of directed edges: equation (5.4.1) of [28] matches the expression for our $C(d, 1, k)$.

From the results on nearest-neighbour (directed) graphs, we may obtain results for nearest-neighbour (undirected) graphs, in which if \mathbf{x} is a nearest neighbour of \mathbf{y} and vice versa, then the edge between \mathbf{x} and \mathbf{y} is counted only once. As an example, we give the following result.

For $d \in \mathbf{N}$ and $\alpha \geq 0$ let $\mathcal{N}^{d,\alpha}(\mathcal{X}_n)$ denote the total power-weighted edge length of the nearest-neighbour (undirected) graph on $\mathcal{X}_n \subset \mathbf{R}^d$. For $d \in \mathbf{N}$, let ω_d be the volume of the union of two unit d -balls with centres unit distance apart in \mathbf{R}^d .

Theorem 3 *Suppose that $d \in \mathbf{N}$, $\alpha \geq 0$ and f satisfies condition (C1). As $n \rightarrow \infty$,*

$$n^{(\alpha-d)/d} \mathcal{N}^{d,\alpha}(\mathcal{X}_n) \xrightarrow{L^2} \Gamma(1 + (\alpha/d)) \left(v_d^{-\alpha/d} - \frac{1}{2} v_d \omega_d^{-1-(\alpha/d)} \right) \int_{\text{supp}(f)} f(\mathbf{x})^{(d-\alpha)/d} d\mathbf{x}. \quad (8)$$

In particular, when $d = 2$ we have, for $\alpha \geq 0$

$$n^{(\alpha-2)/2} \mathcal{N}^{2,\alpha}(\mathcal{U}_n) \xrightarrow{L^2} \Gamma(1 + (\alpha/2)) \left(\pi^{-\alpha/2} - \frac{\pi}{2} \left(\frac{6}{8\pi + 3\sqrt{3}} \right)^{1+(\alpha/2)} \right), \quad (9)$$

and when $d = 2$, $\alpha = 1$, we get

$$n^{-1/2} \mathcal{N}^{2,1}(\mathcal{U}_n) \xrightarrow{L^2} \frac{1}{2} - \frac{1}{4} \left(\frac{6\pi}{8\pi + 3\sqrt{3}} \right)^{3/2} \approx 0.377508. \quad (10)$$

Finally, when $d = 1$, $\alpha = 1$, we have $\mathcal{N}^{1,1}(\mathcal{U}_n) \xrightarrow{L^2} 7/18$ as $n \rightarrow \infty$.

Remark. A pair of points, each of which is the other's nearest neighbour, is known as a reciprocal pair. Reciprocal pairs are of interest in ecology (see [27]). When $\alpha = 0$, $\mathcal{N}^{d,0}(\mathcal{X}_n)$ counts the number of vertices, minus one half of the number of reciprocal pairs.

In this case (8) says $n^{-1} \mathcal{N}^{d,0}(\mathcal{X}_n) \xrightarrow{L^2} 1 - (v_d/(2\omega_d))$. This is consistent with results of Henze [12] for the fraction of points that are the ℓ -th nearest neighbour of their own k -th nearest neighbour; in particular, (see [12] and references therein) as $n \rightarrow \infty$, the probability that a point is in a reciprocal pair tends to v_d/ω_d .

2.2 The on-line nearest-neighbour graph

We now consider the *on-line nearest-neighbour graph* (ONG). Let $d \in \mathbf{N}$. Suppose $\mathbf{x}_1, \mathbf{x}_2, \dots$ are points in $(0, 1)^d$, arriving sequentially; for $n \in \mathbf{N}$ form a graph on vertex set $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ by connecting each point \mathbf{x}_i , $i = 2, 3, \dots, n$ to its nearest neighbour amongst its predecessors (i.e. $\mathbf{x}_1, \dots, \mathbf{x}_{i-1}$), using the lexicographic ordering on \mathbf{R}^d to break any ties. The resulting tree is the ONG on $(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$.

Again, we take our sequence of points to be random. We restrict our analysis to the case in which we have independent uniformly distributed points $\mathbf{U}_1, \mathbf{U}_2, \dots$ on $(0, 1)^d$. For $d \in \mathbf{N}$, $\alpha \geq 0$ and $n \in \mathbf{N}$, let $\mathcal{O}^{d,\alpha}(\mathcal{U}_n)$ denote the total power-weighted edge length of the ONG on sequence $\mathcal{U}_n = (\mathbf{U}_1, \dots, \mathbf{U}_n)$. The next result gives a new LLN for $\mathcal{O}^{d,\alpha}(\mathcal{U}_n)$ when $\alpha < d$.

Theorem 4 Suppose $d \in \mathbf{N}$ and $\alpha \in [0, d)$. With $C(d, \alpha, k)$ as given by (4), we have that as $n \rightarrow \infty$

$$n^{(\alpha-d)/d} \mathcal{O}^{d,\alpha}(\mathcal{U}_n) \xrightarrow{L^1} \frac{d}{d-\alpha} C(d, \alpha, 1) = \frac{d}{d-\alpha} v_d^{-\alpha/d} \Gamma(1 + (\alpha/d)). \quad (11)$$

Related results include those on convergence in distribution of $\mathcal{O}^{d,\alpha}(\mathcal{U}_n)$, given in [24] for $\alpha > d$ ($\alpha > 1/2$ in the case $d = 1$) and in [21] in the form of a central limit theorem for $\alpha \in (0, 1/4)$. Also, the ONG in $d = 1$ is related to the ‘directed linear tree’ considered in [23].

2.3 The minimal directed spanning forest

The *minimal directed spanning forest* (MDSF) is related to the standard nearest-neighbour (directed) graph, with the additional constraint that edges can only lie in a given direction. In general, the MDSF can be defined as a global optimization problem for directed graphs on partially ordered sets endowed with a weight function, and it also admits a local construction; see [5, 22, 23]. As above, we consider the Euclidean setting, where our points lie in \mathbf{R}^d .

Suppose that $\mathcal{X} \subset \mathbf{R}^d$ is a finite set bearing a partial order \preceq . A *minimal element*, or *sink*, of \mathcal{X} is a vertex $\mathbf{v}_0 \in \mathcal{X}$ for which there exists no $\mathbf{v} \in \mathcal{X} \setminus \{\mathbf{v}_0\}$ such that $\mathbf{v} \preceq \mathbf{v}_0$. Let \mathcal{S} denote the set of all sinks of \mathcal{X} . (Note that \mathcal{S} cannot be empty.)

For $\mathbf{v} \in \mathcal{X}$, we say that $\mathbf{u} \in \mathcal{X} \setminus \{\mathbf{v}\}$ is a *directed nearest neighbour* of \mathbf{v} if $\mathbf{u} \preceq \mathbf{v}$ and $\|\mathbf{v} - \mathbf{u}\| \leq \|\mathbf{v} - \mathbf{u}'\|$ for all $\mathbf{u}' \in \mathcal{X} \setminus \{\mathbf{v}\}$ such that $\mathbf{u}' \preceq \mathbf{v}$. For each $\mathbf{v} \in \mathcal{X} \setminus \mathcal{S}$, let $n_{\mathbf{v}}$ be a directed nearest neighbour of \mathbf{v} (chosen arbitrarily if \mathbf{v} has more than one). Then (see [22]) the directed graph on \mathcal{X} obtained by taking edge set $E := \{(\mathbf{v}, n_{\mathbf{v}}) : \mathbf{v} \in \mathcal{X} \setminus \mathcal{S}\}$ is a MDSF of \mathcal{X} . Thus, if all edge-weights are distinct, the MDSF is unique, and is obtained by connecting each non-minimal vertex to its directed nearest neighbour. In the case where there is a single sink, the MDSF is a tree (ignoring directedness of edges) and it is called the *minimal directed spanning tree* (MDST).

For what follows, we consider a general type of partial order on \mathbf{R}^2 , denoted $\preceq^{\theta,\phi}$, specified by the angles $\theta \in [0, 2\pi)$ and $\phi \in (0, \pi]$. For $\mathbf{x} \in \mathbf{R}^2$, let $C_{\theta,\phi}(\mathbf{x})$ be the closed half-cone of angle ϕ with vertex \mathbf{x} and boundaries given by the rays from \mathbf{x} at angles θ and $\theta + \phi$, measuring anticlockwise from the upwards vertical. The partial order is such that, for $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{R}^2$,

$$\mathbf{x}_1 \preceq^{\theta,\phi} \mathbf{x}_2 \text{ iff } \mathbf{x}_1 \in C_{\theta,\phi}(\mathbf{x}_2). \quad (12)$$

We shall use \preceq^* as shorthand for the special case $\preceq^{\pi/2, \pi/2}$, which is of particular interest, as in [5]. In this case $(u_1, u_2) \preceq^* (v_1, v_2)$ iff $u_1 \leq v_1$ and $u_2 \leq v_2$. The symbol \preceq will denote a general partial order on \mathbf{R}^2 . Note that in the case $\phi = \pi$, (12) does not, in fact, define a partial order on the whole of \mathbf{R}^2 , since the antisymmetric property ($\mathbf{x} \preceq \mathbf{y}$ and $\mathbf{y} \preceq \mathbf{x}$ implies $\mathbf{x} = \mathbf{y}$) fails; however it is, with probability one, a true partial order (in fact, a total order) on the random point sets that we consider.

We do not permit here the case $\phi = 0$, which would almost surely give us a disconnected point set. Nor do we allow $\phi \in (\pi, 2\pi]$, since in this case the directional relation (12) is

not a partial order, since the transitivity property (if $\mathbf{u} \preceq \mathbf{v}$ and $\mathbf{v} \preceq \mathbf{w}$ then $\mathbf{u} \preceq \mathbf{w}$) fails for $\phi \in (\pi, 2\pi]$.

Again we take \mathcal{X} to be random; set $\mathcal{X} = \mathcal{X}_n$, where (as before) \mathcal{X}_n is a point process consisting of n independent random points on $(0, 1)^2$ with common density f . When the partial order is \preceq^* , as in [5], we also consider the point set $\mathcal{X}_n^0 := \mathcal{X}_n \cup \{\mathbf{0}\}$ (where $\mathbf{0}$ is the origin in \mathbf{R}^2) on which the MDSF is a MDST rooted at $\mathbf{0}$.

In this random setting, almost surely each point of \mathcal{X} has a unique directed nearest neighbour, so that \mathcal{X} has a unique MDSF. Denote by $\mathcal{M}^\alpha(\mathcal{X})$ the total power-weighted edge length, with weight exponent $\alpha > 0$, of the MDSF on \mathcal{X} .

Theorem 5 presents LLNs for $\mathcal{M}^\alpha(\mathcal{X}_n)$ in the uniform case $\mathcal{X}_n = \mathcal{U}_n$. However, the proof carries through to other distributions. In particular, if the points of \mathcal{X}_n are distributed in \mathbf{R}^2 with a density f that satisfies condition (C1) above, then (13) holds with a factor of $\int_{\text{supp}(f)} f(\mathbf{x})^{(2-\alpha)/2} d\mathbf{x}$ introduced into the right-hand side.

Theorem 5 *Let $d \in \mathbf{N}$ and $\alpha \in (0, 2)$. Under partial order $\preceq^{\theta, \phi}$ with $\theta \in [0, 2\pi)$ and $\phi \in (0, \pi]$, we have that, as $n \rightarrow \infty$,*

$$n^{(\alpha-2)/2} \mathcal{M}^\alpha(\mathcal{U}_n) \xrightarrow{L^1} (2/\phi)^{\alpha/2} \Gamma(1 + (\alpha/2)). \quad (13)$$

Moreover, when the partial order is \preceq^* , (13) remains true with \mathcal{U}_n replaced by \mathcal{U}_n^0 .

2.4 The Gabriel graph

In the *Gabriel graph* (see [11]) on point set $\mathcal{X} \subset \mathbf{R}^d$, two points are joined by an edge iff the ball that has the line segment joining those two points as a diameter contains no other points of \mathcal{X} . The Gabriel graph has been applied in many of the same contexts as nearest-neighbour graphs; see for example [30].

For $d \in \mathbf{N}$ and $\alpha \geq 0$, let $\mathcal{G}^{d, \alpha}(\mathcal{X})$ denote the total power-weighted edge length of the Gabriel graph on $\mathcal{X} \subset \mathbf{R}^d$. As before, we consider the random point set \mathcal{X}_n with underlying density f . A LLN for $\mathcal{G}^{d, \alpha}(\mathcal{X}_n)$ was given in [26]; in the present paper we give the limiting constant explicitly.

Theorem 6 *Let $d \in \mathbf{N}$ and $\alpha \geq 0$. Suppose that f satisfies (C1). As $n \rightarrow \infty$,*

$$n^{(\alpha-d)/d} \mathcal{G}^{d, \alpha}(\mathcal{X}_n) \xrightarrow{L^2} v_d^{-\alpha/d} 2^{d+\alpha-1} \Gamma(1 + (\alpha/d)) \int_{\text{supp}(f)} f(\mathbf{x})^{(d-\alpha)/d} d\mathbf{x}. \quad (14)$$

3 Proofs

3.1 Proof of Theorems 2 and 3

For $j \in \mathbf{N}$, let $d_j(\mathbf{x}; \mathcal{X})$ be the (Euclidean) distance from \mathbf{x} to its j -th nearest neighbour in $\mathcal{X} \setminus \{\mathbf{x}\}$, if such a neighbour exists, or zero otherwise. We will use the following form of Euler's Gamma integral (see equation 6.1.1 in [1]). For $a, b, c \geq 0$

$$\int_0^\infty r^a e^{-cr^b} dr = \frac{1}{b} c^{-(a+1)/b} \Gamma((a+1)/b). \quad (15)$$

Proof of Theorem 2. In applying Theorem 1 to the j -th NNG' and k -NNG' functionals, we take $\xi(\mathbf{x}; \mathcal{X}_n)$ to be $(d_j(\mathbf{x}; \mathcal{X}_n))^\alpha$, where $\alpha \geq 0$. Then ξ is translation invariant and homogeneous of order α . It was shown in Theorem 2.4 of [26] that the j -th NNG' total weight functional ξ satisfies the conditions of Theorem 1 in the following two cases: (i) with $q = 2$, if f satisfies (C1), and $\alpha \geq 0$; and (ii) with $q = 1$, if f satisfies (C2), and $0 \leq \alpha < d$. (In fact, in [26] this is proved for the k -NNG' functional $\sum_{j=1}^k (d_j(\mathbf{x}; \mathcal{X}_n))^\alpha$, but this implies that the conditions also hold for the j -th NNG' functional $(d_j(\mathbf{x}; \mathcal{X}_n))^\alpha$.)

The functional $\xi(\mathbf{x}; \mathcal{X}_n) = (d_j(\mathbf{x}; \mathcal{X}_n))^\alpha$ stabilizes on \mathcal{H}_1 , with limit $\xi_\infty(\mathcal{H}_1) = (d_j(\mathbf{0}; \mathcal{H}_1))^\alpha$. Also, the moment condition (1) is satisfied for some $p > 1$ (if f satisfies (C2) and $\alpha < d$) or $p > 2$ (if f satisfies (C1)), and so Theorem 1, with $q = 1$ or $q = 2$ respectively, yields (using the fact that ξ is homogeneous of order α)

$$\begin{aligned} n^{(\alpha/d)-1} \mathcal{L}_j^{d,\alpha}(\mathcal{X}_n) &= n^{-1} \sum_{\mathbf{x} \in \mathcal{X}_n} n^{\alpha/d} \xi(\mathbf{x}; \mathcal{X}_n) = n^{-1} \sum_{\mathbf{x} \in \mathcal{X}_n} \xi(n^{1/d} \mathbf{x}; n^{1/d} \mathcal{X}_n) \\ &\xrightarrow{L^q} E[\xi_\infty(\mathcal{H}_1)] \int_{\text{supp}(f)} f(\mathbf{x})^{(d-\alpha)/d} d\mathbf{x}. \end{aligned} \quad (16)$$

We now need to evaluate the expectation on the right-hand side of (16). For $r > 0$

$$\begin{aligned} P[\xi_\infty(\mathcal{H}_1) > r] &= P[d_j(\mathbf{0}; \mathcal{H}_1) > r^{1/\alpha}] = \sum_{i=0}^{j-1} P[\text{card}(\{B(\mathbf{0}; r^{1/\alpha}) \cap \mathcal{H}_1\}) = i] \\ &= \sum_{i=0}^{j-1} \frac{(v_d r^{d/\alpha})^i}{i!} \exp(-v_d r^{d/\alpha}), \end{aligned}$$

where v_d is given by (3). So

$$E[\xi_\infty(\mathcal{H}_1)] = \int_0^\infty P[\xi_\infty(\mathcal{H}_1) > r] dr = \int_0^\infty \sum_{i=0}^{j-1} \frac{(v_d r^{d/\alpha})^i}{i!} \exp(-v_d r^{d/\alpha}) dr.$$

Interchanging the order of summation and integration, and using (15), we obtain

$$E[\xi_\infty(\mathcal{H}_1)] = v_d^{-\alpha/d} \frac{\alpha}{d} \sum_{i=0}^{j-1} \frac{\Gamma(i + (\alpha/d))}{\Gamma(i+1)} = v_d^{-\alpha/d} \frac{\Gamma(j + (\alpha/d))}{\Gamma(j)}, \quad (17)$$

where the final equality follows by induction on j . Then from (3), (16) and (17) we obtain the j -th NNG' result (5). By (2), the k -NNG' result (6) follows from (5) with

$$C(d, \alpha, k) = v_d^{-\alpha/d} \sum_{j=1}^k \frac{\Gamma(j + (\alpha/d))}{\Gamma(j)} = v_d^{-\alpha/d} \frac{d}{d + \alpha} \frac{\Gamma(k + 1 + (\alpha/d))}{\Gamma(k)}. \quad \square$$

Proof of Theorem 3. The nearest-neighbour (directed) graph counts the weights of edges from points that are nearest neighbours of their own nearest neighbours twice, while the nearest-neighbour (undirected) graph counts such weights only once.

Let $q(\mathbf{x}; \mathcal{X})$ be the distance from \mathbf{x} to its nearest neighbour in $\mathcal{X} \setminus \{\mathbf{x}\}$ if \mathbf{x} is a nearest neighbour of its own nearest neighbour, and zero otherwise. Recall that $d_1(\mathbf{x}; \mathcal{X})$ is the distance from \mathbf{x} to its nearest neighbour in $\mathcal{X} \setminus \{\mathbf{x}\}$. For $\alpha \geq 0$, define

$$\xi'(\mathbf{x}; \mathcal{X}) := (d_1(\mathbf{x}; \mathcal{X}))^\alpha - \frac{1}{2}(q(\mathbf{x}; \mathcal{X}))^\alpha.$$

Then $\sum_{\mathbf{x} \in \mathcal{X}} \xi'(\mathbf{x}, \mathcal{X})$ is the total weight of the nearest-neighbour (undirected) graph on \mathcal{X} . Note that ξ' is translation invariant and homogeneous of order α .

One can check that ξ' is stabilizing on the Poisson process \mathcal{H}_1 , using similar arguments to those for the j -th NNG' and k -NNG' functionals. Also (see [26]) if condition (C1) holds then ξ' satisfies the moments condition (1) for some $p > 2$, for all $\alpha \geq 0$.

Let \mathbf{e}_1 be a vector of unit length in \mathbf{R}^d . For $d \in \mathbf{N}$, let $\omega_d := |B(\mathbf{0}; 1) \cup B(\mathbf{e}_1; 1)|$, the volume of the union of two unit d -balls with centres unit distance apart.

Now we apply Theorem 1 with $q = 2$. We have

$$\begin{aligned} n^{(\alpha/d)-1} \mathcal{N}^{d,\alpha}(\mathcal{X}_n) &= n^{-1} \sum_{\mathbf{x} \in \mathcal{X}_n} \xi'(n^{1/d} \mathbf{x}; n^{1/d} \mathcal{X}_n) \\ &\xrightarrow{L^2} E[\xi'_\infty(\mathcal{H}_1)] \int_{\text{supp}(f)} f(\mathbf{x})^{(d-\alpha)/d} d\mathbf{x}, \end{aligned} \quad (18)$$

where $E[\xi'_\infty(\mathcal{H}_1)] = E[(d_1(\mathbf{0}; \mathcal{H}_1))^\alpha] - (1/2)E[(q(\mathbf{0}; \mathcal{H}_1))^\alpha]$. Now we need to evaluate $E[(q(\mathbf{0}; \mathcal{H}_1))^\alpha]$. With \mathbf{X} denoting the nearest point of \mathcal{H}_1 to $\mathbf{0}$,

$$\begin{aligned} P[q(\mathbf{0}; \mathcal{H}_1) \in dr] &= P[\{|\mathbf{X}| \in dr\} \cap \{\mathcal{H}_1 \cap (B(\mathbf{0}; r) \cup B(\mathbf{X}; r)) = \{\mathbf{X}\}\}] \\ &= dv_d r^{d-1} e^{-v_d r^d} e^{-(\omega_d - v_d) r^d} dr = dv_d r^{d-1} e^{-\omega_d r^d} dr. \end{aligned}$$

So using (15) we obtain

$$E[(q(\mathbf{0}; \mathcal{H}_1))^\alpha] = \int_0^\infty dv_d r^{d-1+\alpha} e^{-\omega_d r^d} dr = v_d \omega_d^{-1-(\alpha/d)} \Gamma(1 + (\alpha/d)). \quad (19)$$

Then from (18) with (19) and the $j = 1$ case of (17) we obtain (8). By some calculus, $\omega_2 = (4\pi/3) + (\sqrt{3}/2)$, which with the $d = 2$ case of (8) yields (9); for (10) note that $\Gamma(3/2) = \pi^{1/2}/2$ (see 6.1.9 in [1]). Finally, we obtain the statement for $\mathcal{N}^{1,1}(\mathcal{U}_n)$ from the $d = 1$ case of (8) since $\omega_1 = 3$. \square

3.2 Proof of Theorem 4

In order to obtain our LLN (Theorem 4 above), we modify the setup of the ONG slightly. Let \mathcal{U}_n be a *marked* random finite point process in \mathbf{R}^d , consisting of n independent uniform random vectors in $(0, 1)^d$, where each point \mathbf{U}_i of \mathcal{U}_n carries a random mark $T(\mathbf{U}_i)$ which is uniformly distributed on $[0, 1]$, independent of the other marks and of the point process \mathcal{U}_n . Join each point \mathbf{U}_i of \mathcal{U}_n to its nearest neighbour amongst those points of \mathcal{U}_n with mark less than $T(\mathbf{U}_i)$, if there are any such points, to obtain a graph that we call the ONG on the marked point set \mathcal{U}_n . This definition extends to infinite but locally finite point sets.

Clearly the ONG on the marked point process \mathcal{U}_n has the same distribution as the ONG (with the first definition) on a sequence $\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n$ of independent uniform points on $(0, 1)^d$.

We apply Theorem 1 to obtain a LLN for $\mathcal{O}^{d,\alpha}(\mathcal{U}_n)$, $\alpha \in [0, d]$. Once again, the method enables us to evaluate the limit explicitly. We take f to be the indicator of $(0, 1)^d$. Define $D(\mathbf{x}; \mathcal{X})$ to be the distance from point \mathbf{x} with mark $T(\mathbf{x})$ to its nearest neighbour in \mathcal{X} amongst those points $\mathbf{y} \in \mathcal{X}$ that have mark $T(\mathbf{y})$ such that $T(\mathbf{y}) < T(\mathbf{x})$, if such a neighbour exists, or zero otherwise. We take $\xi(\mathbf{x}; \mathcal{X})$ to be $(D(\mathbf{x}; \mathcal{X}))^\alpha$. Again, ξ is translation invariant and homogeneous of order α .

Lemma 1 *The ONG functional ξ almost surely stabilizes on \mathcal{H}_1 .*

Proof. Although the notion of stabilization there is somewhat different, the same argument as given at the start of the proof of Theorem 3.6 of [21] applies. \square

Lemma 2 *Let $d \in \mathbf{N}$, $\alpha \in [0, d)$, and let $p > 1$ with $\alpha p < d$. Then the ONG functional ξ satisfies the moments condition (1).*

Proof. Let T_n denote the rank of the mark of \mathbf{U}_1 amongst the marks of all the points of \mathcal{U}_n , so that T_n is distributed uniformly over the integers $1, 2, \dots, n$. We have, by conditioning on T_n ,

$$\begin{aligned} E[(\xi(n^{1/d}\mathbf{U}_1; n^{1/d}\mathcal{U}_n))^p] &= n^{-1} \sum_{i=1}^n E[(d_1(n^{1/d}\mathbf{U}_1; n^{1/d}\mathcal{U}_i))^{p\alpha}] \\ &= n^{-1} \sum_{i=1}^n (n/i)^{p\alpha/d} E[(d_1(i^{1/d}\mathbf{U}_1; i^{1/d}\mathcal{U}_i))^{p\alpha}]. \end{aligned} \quad (20)$$

It was shown in [24] that there exists $C \in (0, \infty)$ such that for all $r > 0$

$$\sup_{i \geq 1} P[d_1(i^{1/d}\mathbf{U}_1; i^{1/d}\mathcal{U}_i) > r] \leq C \exp(-r^{1/d}/C).$$

Thus the last expectation in (20) is bounded by a constant independent of i . So the final expression in (20) is bounded by a constant times

$$n^{(p\alpha-d)/d} \sum_{i=1}^n i^{-p\alpha/d},$$

which is uniformly bounded by a constant for $\alpha p < d$. \square

Proof of Theorem 4. Let $d \in \mathbf{N}$. Let f be the indicator of $(0, 1)^d$, and ξ be the ONG functional $\xi(\mathbf{x}; \mathcal{U}_n) = (D(\mathbf{x}; \mathcal{U}_n))^\alpha$. By Lemmas 1 and 2, ξ is homogeneous of order α , stabilizing on \mathcal{H}_1 with limit $\xi_\infty(\mathcal{H}_1) = (D(\mathbf{0}; \mathcal{H}_1))^\alpha$, and satisfies the moment condition (1) for some $p > 1$, provided $\alpha < d$. So Theorem 1 with $q = 1$ implies

$$n^{(\alpha/d)-1} \mathcal{O}^{d,\alpha}(\mathcal{U}_n) = n^{-1} \sum_{\mathbf{x} \in \mathcal{U}_n} (D(n^{1/d}\mathbf{x}; n^{1/d}\mathcal{U}_n))^\alpha \xrightarrow{L^1} E[\xi_\infty(\mathcal{H}_1)].$$

For $u \in (0, 1)$ the points of \mathcal{H}_1 with lower mark than u form a homogeneous Poisson point process of intensity u , so by conditioning on the mark of the point at $\mathbf{0}$,

$$E[\xi_\infty(\mathcal{H}_1)] = \int_0^1 E[(d_1(\mathbf{0}; \mathcal{H}_u))^\alpha] du = \int_0^1 u^{-\alpha/d} E[(d_1(\mathbf{0}; \mathcal{H}_1))^\alpha] du = \frac{d}{d-\alpha} C(d, \alpha, 1),$$

since we saw in the proof of Theorem 2 that $E[(d_1(\mathbf{0}; \mathcal{H}_1))^\alpha] = C(d, \alpha, 1)$. \square

3.3 Proof of Theorem 5

In applying Theorem 1 to the MDSF, we take f to be the indicator of $(0, 1)^2$. We take $\xi(\mathbf{x}; \mathcal{X})$ to be $(d(\mathbf{x}; \mathcal{X}))^\alpha$, where $d(\mathbf{x}; \mathcal{X})$ is the distance from point \mathbf{x} to its directed nearest neighbour in $\mathcal{X} \setminus \{\mathbf{x}\}$, if such a point exists, or zero otherwise, i.e.

$$\xi(\mathbf{x}; \mathcal{X}) = (d(\mathbf{x}; \mathcal{X}))^\alpha \quad \text{with} \quad d(\mathbf{x}; \mathcal{X}) := \min\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{y} \in \mathcal{X} \setminus \{\mathbf{x}\}, \mathbf{y} \stackrel{\theta, \phi}{\preceq} \mathbf{x}\} \quad (21)$$

with the convention that $\min \emptyset = 0$.

We consider the random point set \mathcal{U}_n , the binomial point process consisting of n independent uniformly distributed points on $(0, 1)^2$. However, as remarked before the statement of Theorem 5, the result (13) carries through (with virtually the same proof) to more general point sets \mathcal{X}_n .

We need to show that ξ given by (21) satisfies the conditions of Theorem 1. As before, \mathcal{H}_1 denotes a homogeneous Poisson process on \mathbf{R}^2 .

Lemma 3 *The MDSF functional ξ given by (21) almost surely stabilizes on \mathcal{H}_1 with limit $\xi_\infty(\mathcal{H}_1) = (d(\mathbf{0}; \mathcal{H}_1))^\alpha$.*

Proof. Set $R := d(\mathbf{0}; \mathcal{H}_1)$. Since $\phi > 0$ we have $0 < R < \infty$ almost surely. But then for any $\ell > R$, we have $\xi(\mathbf{0}; (\mathcal{H}_1 \cap B(\mathbf{0}; \ell)) \cup \mathcal{A}) = R^\alpha$, for any finite $\mathcal{A} \subset \mathbf{R}^d \setminus B(\mathbf{0}; \ell)$. Thus ξ stabilizes on \mathcal{H}_1 with limit $\xi_\infty(\mathcal{H}_1) = R^\alpha$. \square

We now give a geometrical lemma. For $B \subset \mathbf{R}^2$ with B bounded, and for $\mathbf{x} \in B$, write $\text{dist}(\mathbf{x}; \partial B)$ for $\sup\{r : B(\mathbf{x}; r) \subseteq B\}$, and for $s > 0$, define the region

$$A_{\theta, \phi}(\mathbf{x}, s; B) := B(\mathbf{x}; s) \cap B \cap C_{\theta, \phi}(\mathbf{x}). \quad (22)$$

Lemma 4 *Let B be a convex bounded set in \mathbf{R}^2 , and let $\mathbf{x} \in B$. If $A_{\theta, \phi}(\mathbf{x}, s; B) \cap \partial B(\mathbf{x}; s) \neq \emptyset$, and $s > \text{dist}(\mathbf{x}, \partial B)$, then*

$$|A_{\theta, \phi}(\mathbf{x}, s; B)| \geq s \sin(\phi/2) \text{dist}(\mathbf{x}, \partial B)/2.$$

Proof. The condition $A_{\theta, \phi}(\mathbf{x}, s; B) \cap \partial B(\mathbf{x}; s) \neq \emptyset$ says that there exists $\mathbf{y} \in B \cap C_{\theta, \phi}(\mathbf{x})$ with $\|\mathbf{y} - \mathbf{x}\| = s$. The line segment \mathbf{xy} is contained in the cone $C_{\theta, \phi}(\mathbf{x})$; take a half-line \mathbf{h} starting from \mathbf{x} , at an angle $\phi/2$ to the line segment \mathbf{xy} and such that \mathbf{h} is also contained in $C_{\theta, \phi}(\mathbf{x})$. Let \mathbf{z} be the point in \mathbf{h} at a distance $\text{dist}(\mathbf{x}, \partial B)$ from \mathbf{x} . Then the interior of the triangle \mathbf{xyz} is entirely contained in $A_{\theta, \phi}(\mathbf{x}, s)$, and has area $s \sin(\phi/2) \text{dist}(\mathbf{x}, \partial B)/2$. \square

Lemma 5 *Suppose $\alpha > 0$. Then the MDSF functional ξ given by (21) satisfies the moments condition (1) for any $p \leq 2/\alpha$.*

Proof. Setting $R_n := (0, n^{1/2})^2$, conditioning on the position of \mathbf{U}_1 , we have

$$E[\xi(n^{1/2}\mathbf{U}_1; n^{1/2}\mathcal{U}_n)^p] = n^{-1} \int_{R_n} E[(\xi(\mathbf{x}; n^{1/2}\mathcal{U}_{n-1}))^p] d\mathbf{x}. \quad (23)$$

For $\mathbf{x} \in R_n$ set $m(\mathbf{x}) := \text{dist}(\mathbf{x}, \partial R_n)$. We divide R_n into three regions

$$\begin{aligned} R_n(1) &:= \{\mathbf{x} \in R_n : m(\mathbf{x}) \leq n^{-1/2}\}; & R_n(2) &:= \{\mathbf{x} \in R_n : m(\mathbf{x}) > 1\}; \\ R_n(3) &:= \{\mathbf{x} \in R_n : n^{-1/2} < m(\mathbf{x}) \leq 1\}. \end{aligned}$$

For all $\mathbf{x} \in R_n$, we have $\xi(\mathbf{x}; n^{1/2}\mathcal{U}_{n-1}) \leq (2n)^{\alpha/2}$, and hence, since $R_n(1)$ has area at most 4, we can bound the contribution to (23) from $\mathbf{x} \in R_n(1)$ by

$$n^{-1} \int_{R_n(1)} E[(\xi(\mathbf{x}; n^{1/2}\mathcal{U}_{n-1}))^p] d\mathbf{x} \leq 4n^{-1}(2n)^{p\alpha/2} = 2^{2+p\alpha/2}n^{(p\alpha-2)/2}, \quad (24)$$

which is bounded if $p\alpha \leq 2$. Now, for $\mathbf{x} \in R_n$, with $A_{\theta,\phi}$ defined at (22), we have

$$\begin{aligned} P[d(\mathbf{x}; n^{1/2}\mathcal{U}_{n-1}) > s] &\leq P[n^{1/2}\mathcal{U}_{n-1} \cap A_{\theta,\phi}(\mathbf{x}, s; R_n) = \emptyset] \\ &= \left(1 - \frac{|A_{\theta,\phi}(\mathbf{x}, s; R_n)|}{n}\right)^{n-1} \\ &\leq \exp(1 - |A_{\theta,\phi}(\mathbf{x}, s; R_n)|), \end{aligned} \quad (25)$$

since $|A_{\theta,\phi}(\mathbf{x}, s; R_n)| \leq n$. For $\mathbf{x} \in R_n$ and $s > m(\mathbf{x})$, by Lemma 4 we have

$$|A_{\theta,\phi}(\mathbf{x}, s; R_n)| \geq s \sin(\phi/2)m(\mathbf{x})/2 \quad \text{if } A_{\theta,\phi}(\mathbf{x}, s; R_n) \cap \partial B(\mathbf{x}; s) \neq \emptyset,$$

and also

$$P[d(\mathbf{x}; n^{1/2}\mathcal{U}_{n-1}) > s] = 0 \quad \text{if } A_{\theta,\phi}(\mathbf{x}, s; R_n) \cap \partial B(\mathbf{x}; s) = \emptyset.$$

For $s \leq m(\mathbf{x})$, we have that $|A_{\theta,\phi}(\mathbf{x}, s; R_n)| = s^2(\phi/2) \geq s^2 \sin(\phi/2)$. Combining these observations and (25), we obtain for all $\mathbf{x} \in R_n$ and $s > 0$ that

$$P[d(\mathbf{x}; n^{1/2}\mathcal{U}_{n-1}) > s] \leq \exp(1 - (s/2) \min(s, m(\mathbf{x})) \sin(\phi/2)), \quad \mathbf{x} \in R_n.$$

Setting $c = (1/2) \sin(\phi/2)$, we therefore have for $\mathbf{x} \in R_n$ that

$$\begin{aligned} E[(\xi(\mathbf{x}; n^{1/2}\mathcal{U}_{n-1}))^p] &= \int_0^\infty P[d(\mathbf{x}; n^{1/2}\mathcal{U}_{n-1}) > r^{1/(\alpha p)}] dr \\ &\leq \int_0^{m(\mathbf{x})^{\alpha p}} \exp(1 - cr^{2/(\alpha p)}) dr + \int_{m(\mathbf{x})^{\alpha p}}^\infty \exp(1 - cm(\mathbf{x})r^{1/(\alpha p)}) dr \\ &= O(1) + \alpha p m(\mathbf{x})^{-p\alpha} \int_{m(\mathbf{x})^2}^\infty e^{1-cu} \alpha p u^{\alpha p-1} du = O(1) + O(m(\mathbf{x})^{-\alpha p}). \end{aligned} \quad (26)$$

For $\mathbf{x} \in R_n(2)$, this bound is $O(1)$, and the area of $R_n(2)$ is less than n , so that the contribution to (23) from $R_n(2)$ satisfies

$$\limsup_{n \rightarrow \infty} n^{-1} \int_{R_n(2)} E[(\xi(\mathbf{x}; n^{1/2}\mathcal{U}_{n-1}))^p] d\mathbf{x} < \infty. \quad (27)$$

Finally, by (26), there is a constant $C \in (0, \infty)$ such that the contribution to (23) from $R_n(3)$ satisfies

$$\begin{aligned} n^{-1} \int_{R_n(3)} E[(\xi(\mathbf{x}; n^{1/2}\mathcal{U}_{n-1}))^p] d\mathbf{x} &\leq Cn^{-1/2} \int_{y=n^{-1/2}}^1 y^{-\alpha p} dy \\ &\leq Cn^{-1/2} \max\{\log n, n^{(\alpha p-1)/2}\}, \end{aligned}$$

which is bounded if $\alpha p \leq 2$. Combined with the bounds in (24) and (27), this shows that the expression (23) is uniformly bounded, provided $\alpha p \leq 2$. \square

For $k \in \mathbf{N}$, and for $a < b$ and $c < d$ let $\mathcal{U}_{k,(a,b] \times (c,d]}$ denote the point process consisting of k independent random vectors uniformly distributed on the rectangle $(a, b] \times (c, d]$. Before proceeding further, we recall that if $M(\mathcal{X})$ denotes the number of minimal elements, under partial order \preceq^* , of a point set $\mathcal{X} \subset \mathbf{R}^2$, then

$$E[M(\mathcal{U}_{k,(a,b] \times (c,d]})] = E[M(\mathcal{U}_k)] = 1 + (1/2) + \cdots + (1/k) \leq 1 + \log k. \quad (28)$$

The first equality in (28) comes from some obvious scaling which shows that the distribution of $M(\mathcal{U}_{k,(a,b] \times (c,d]})$ does not depend on a, b, c, d . For the second equality in (28), see e.g. [3].

Proof of Theorem 5. Suppose $\alpha \in (0, 2)$, and set f to be the indicator of $(0, 1)^2$. By Lemmas 3 and 5 the functional ξ , given at (21), satisfies the conditions of Theorem 1 with $p = 2/\alpha$ and $q = 1$. So by Theorem 1, we have

$$n^{(\alpha/2)-1} \mathcal{M}^\alpha(\mathcal{U}_n) = n^{-1} \sum_{\mathbf{x} \in \mathcal{U}_n} \xi(n^{1/2} \mathbf{x}; n^{1/2} \mathcal{U}_n) \xrightarrow{L^1} E[\xi_\infty(\mathcal{H}_1)]. \quad (29)$$

Since the disk sector $C_{\theta,\phi}(\mathbf{x}) \cap B(\mathbf{x}; r)$ has area $(\phi/2)r^2$, by Lemma 3 we have

$$P[\xi_\infty(\mathcal{H}_1) > s] = P[\mathcal{H}_1 \cap C_{\theta,\phi}(\mathbf{0}) \cap B(\mathbf{0}; s^{1/\alpha}) = \emptyset] = \exp(-(\phi/2)s^{2/\alpha}).$$

Hence the limit in (29) is, using (15),

$$E[\xi_\infty(\mathcal{H}_1)] = \int_0^\infty P[\xi_\infty(\mathcal{H}_1) > s] ds = \alpha 2^{(\alpha-2)/2} \phi^{-\alpha/2} \Gamma(\alpha/2),$$

and this gives us (13). Finally, in the case where $\preceq^{\theta,\phi}$ is \preceq^* , (13) remains true when \mathcal{U}_n is replaced by \mathcal{U}_n^0 , since

$$E[n^{(\alpha/2)-1} |\mathcal{M}^\alpha(\mathcal{U}_n^0) - \mathcal{M}^\alpha(\mathcal{U}_n)|] \leq 2^{\alpha/2} n^{(\alpha/2)-1} E[M(\mathcal{U}_n)], \quad (30)$$

where $M(\mathcal{U}_n)$ denotes the number of \preceq^* -minimal elements of \mathcal{U}_n . By (28), $E[M(\mathcal{U}_n)] \leq 1 + \log n$, and hence the right-hand side of (30) tends to 0 as $n \rightarrow \infty$ for $\alpha < 2$. \square

3.4 Proof of Theorem 6

Proof of Theorem 6. In applying Theorem 1 to the Gabriel graph, we take $\xi(\mathbf{x}; \mathcal{X}_n)$ to be *one half* of the total α power-weighted length of all the edges incident to \mathbf{x} in the Gabriel graph on $\mathcal{X}_n \cup \{\mathbf{x}\}$; the factor of one half prevents double counting. As stated in [26] (Section 2.3(e)), ξ is translation invariant, homogeneous of order α and stabilizing on \mathcal{H}_1 , and if the function f satisfies condition (C1) then the moment condition (1) is satisfied for some $p > 2$. So by Theorem 1 with $q = 2$,

$$\begin{aligned} n^{(\alpha/d)-1} \mathcal{G}^{d,\alpha}(\mathcal{X}_n) &= n^{-1} \sum_{\mathbf{x} \in \mathcal{X}_n} \xi(n^{1/d} \mathbf{x}; n^{1/d} \mathcal{X}_n) \\ &\xrightarrow{L^2} E[\xi_\infty(\mathcal{H}_1)] \int_{\text{supp}(f)} f(\mathbf{x})^{(d-\alpha)/d} d\mathbf{x}. \end{aligned} \quad (31)$$

We need to evaluate the expectation on the right-hand side of (31). The net contribution from a vertex at $\mathbf{0}$ to the total weight of the Gabriel graph on \mathcal{H}_1 is

$$\frac{1}{2} \sum_{k=1}^{\infty} (d_k(\mathbf{0}; \mathcal{H}_1))^\alpha \cdot \mathbf{1}_{E_k}, \quad (32)$$

where the factor of one half ensures that edges are not counted twice, $d_k(\mathbf{0}; \mathcal{H}_1)$ is the distance from $\mathbf{0}$ to its k -th nearest neighbour in \mathcal{H}_1 , and E_k denotes the event that $\mathbf{0}$ and its k -th nearest neighbour in \mathcal{H}_1 are joined by an edge in the Gabriel graph.

Given that the point $\mathbf{x} \in \mathcal{H}_1$ is the k -th nearest neighbour of $\mathbf{0}$, an edge between \mathbf{x} and $\mathbf{0}$ exists in the Gabriel graph iff the ball with $\mathbf{0}$ and \mathbf{x} diametrically opposed contains none of the other $k - 1$ points of \mathcal{H}_1 that are uniformly distributed in the ball centre $\mathbf{0}$ and radius $\|\mathbf{x}\|$. Thus for $k \in \mathbf{N}$,

$$P[E_k] = \left(\frac{v_d r^d - v_d (r/2)^d}{v_d r^d} \right)^{k-1} = (1 - 2^{-d})^{k-1}. \quad (33)$$

So from (32) and (33) we have

$$\begin{aligned} E[\xi_\infty(\mathcal{H}_1)] &= \frac{1}{2} \sum_{k=1}^{\infty} (1 - 2^{-d})^{k-1} E[(d_k(\mathbf{0}; \mathcal{H}_1))^\alpha] \\ &= \frac{1}{2} \sum_{k=1}^{\infty} (1 - 2^{-d})^{k-1} v_d^{-\alpha/d} \frac{\Gamma(k + (\alpha/d))}{\Gamma(k)}, \end{aligned}$$

by (17). But by properties of Gauss hypergeometric series (see 15.1.1 and 15.1.8 of [1])

$$\sum_{k=1}^{\infty} (1 - 2^{-d})^{k-1} \frac{\Gamma(k + (\alpha/d))}{\Gamma(k)} = \Gamma(1 + (\alpha/d)) 2^{d+\alpha}.$$

Then with (31) the proof is complete. \square

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